Modeling Nonlinear Relationships among Latent Variables via Mixtures of Linear Structural Equations

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Modeling Nonlinear Effects

- Much attention given to modeling nonlinear effects in SEMs since Kenny & Judd (1984)
  - Product-indicant approaches
  - Method of moment approaches
  - ML or QML approaches
  - Bayesian approaches
- Focus has been on modeling nonlinear effects via known, low-order polynomial functions
- What if functional form is unknown?

A Semiparametric Approach

- Bauer (2005) proposed a semiparametric modeling approach
  - Fit a mixture of linear structural equation models
  - Aggregate over components of the mixture to obtain smooth approximation of nonlinear function
  - Model is locally linear (within mixing component), but globally nonlinear (across mixing components)

The SEM

The SEM (y-side form) consists of…

measurement model:

$$\mathbf{y}_i = \boldsymbol{\nu} + \boldsymbol{\Lambda}\boldsymbol{\eta}_i + \boldsymbol{\varepsilon}_i; \quad \boldsymbol{\varepsilon}_i \sim N_p(\mathbf{0}, \boldsymbol{\Theta})$$

linear structural model:

$$\boldsymbol{\eta}_i = \boldsymbol{\alpha} + \mathbf{B}\boldsymbol{\eta}_i + \boldsymbol{\zeta}_i; \quad \boldsymbol{\zeta}_i \sim N_q(\mathbf{0}, \boldsymbol{\Psi})$$

The SEM

- By implication, marginal distribution of $\mathbf{y}$ is normal

$$\phi[\mathbf{y};\ \boldsymbol{\mu}(\boldsymbol{\theta}),\ \boldsymbol{\Sigma}(\boldsymbol{\theta})]$$

with implied mean and covariance structure

$$\boldsymbol{\mu}(\boldsymbol{\theta}) = \boldsymbol{\nu} + \boldsymbol{\Lambda}(\mathbf{I}-\mathbf{B})^{-1}\boldsymbol{\alpha}$$

$$\boldsymbol{\Sigma}(\boldsymbol{\theta}) = \boldsymbol{\Lambda}(\mathbf{I}-\mathbf{B})^{-1}\boldsymbol{\Psi}(\mathbf{I}-\mathbf{B})^{-1'}\boldsymbol{\Lambda}' + \boldsymbol{\Theta}$$

Finite Mixture SEM

- Finite mixture SEM involves fitting SEMs within each of K latent components

$$f(\mathbf{y}) = \sum_{k=1}^{K} P(k)\phi_k\left[\mathbf{y};\ \boldsymbol{\mu}_k(\boldsymbol{\theta}_k), \boldsymbol{\Sigma}_k(\boldsymbol{\theta}_k)\right]$$

- $P(k)$ are mixing probabilities
- K usually not estimated; chosen through model comparison
- Parameters can vary over components or be constrained to equality

Mixture Applications

- Direct:
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- Indirect:
  - Goal is to use mixture to capture data features not easily recovered using other distributions
  - Component distributions are only a statistical expedience
  - Interpretation focuses on the aggregate mixture distribution

Nonlinear Effect Modeling by FM-SEM

- Locally linear within component:

$$E_k(\boldsymbol{\eta}_2 \mid \boldsymbol{\eta}_1) = \boldsymbol{\alpha}_{2k} + \mathbf{B}_{21k}\boldsymbol{\eta}_1$$

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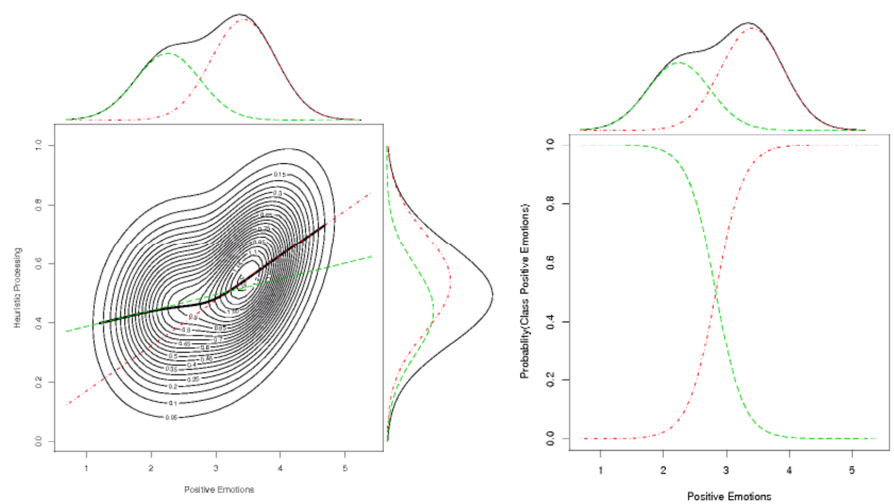
- Smoothing weights are conditional probabilities:

$$P(k \mid \boldsymbol{\eta}_1) = \frac{P(k)\phi_k(\boldsymbol{\eta}_1; \boldsymbol{\alpha}_{1k}, \boldsymbol{\Psi}_{11k})}{\sum_{k=1}^{K} P(k)\phi_k(\boldsymbol{\eta}_1; \boldsymbol{\alpha}_{1k}, \boldsymbol{\Psi}_{11k})}$$

Bivariate Example



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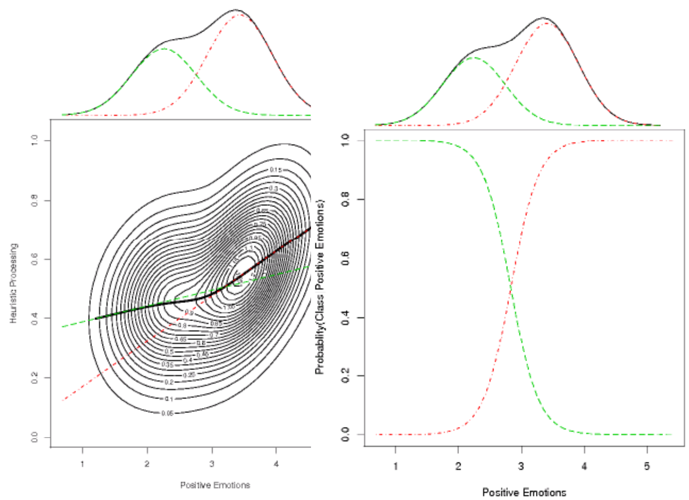
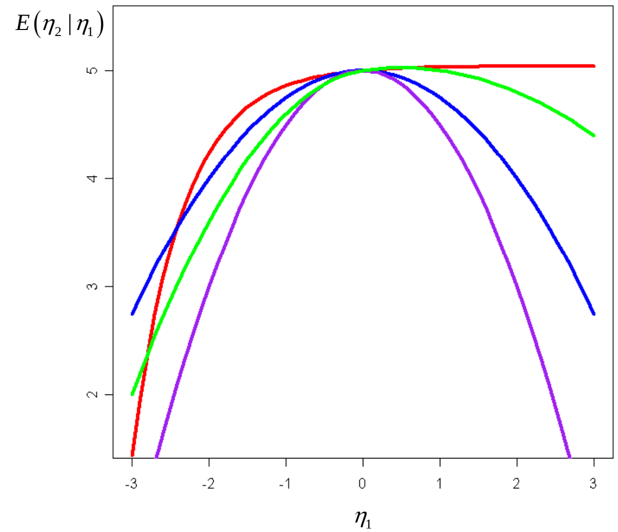
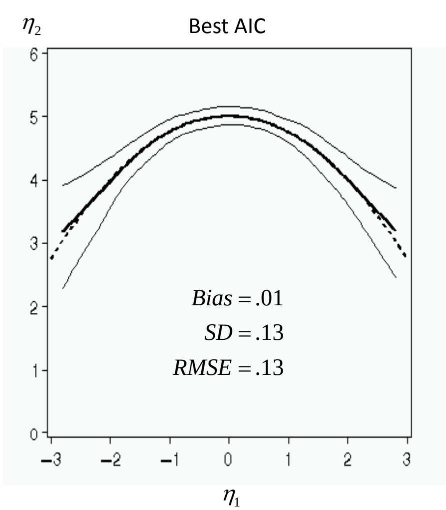
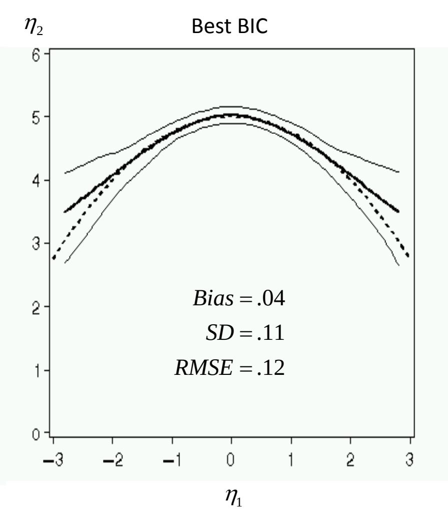


Does it Work?

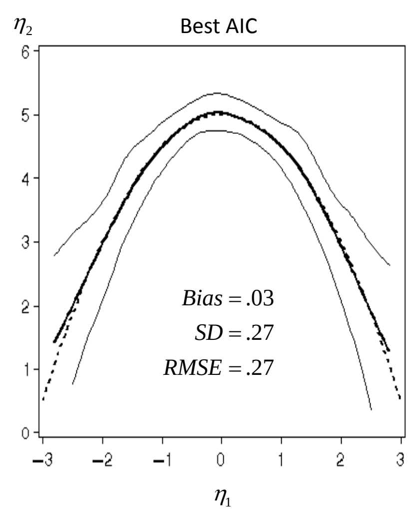


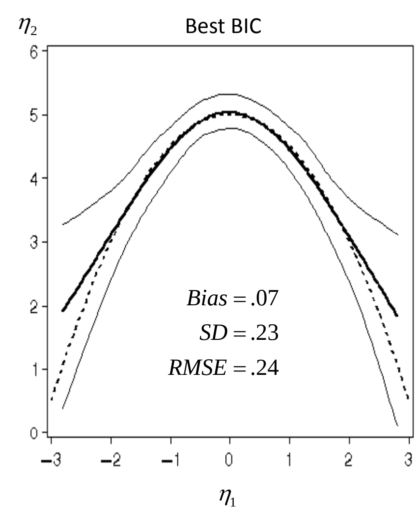
Small Quadratic



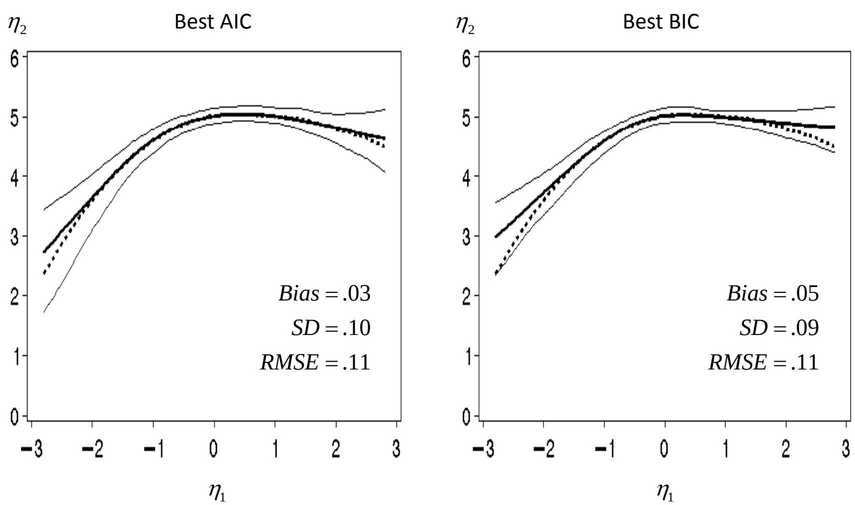


Large Quadratic

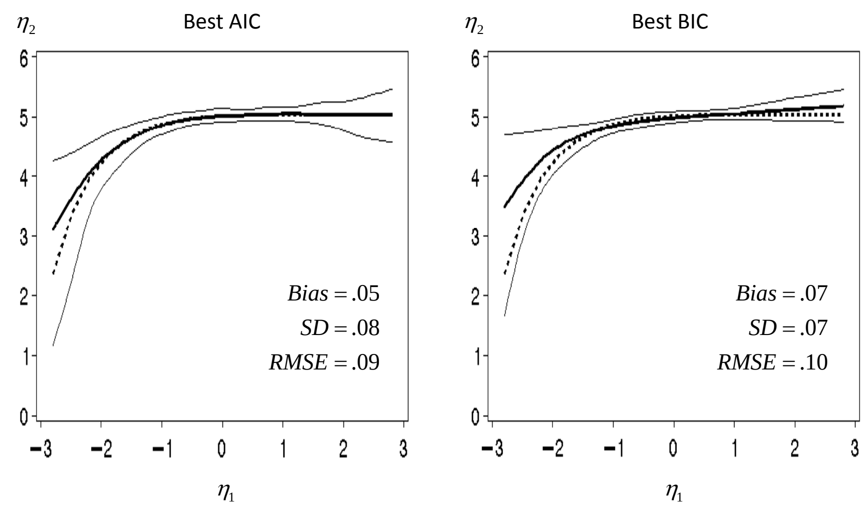




Quadratic Spline



Exponential



Is it Worth the Trouble?

- Why not just plot factor score estimates instead?
- Compared the FM-SEM approach to a three-step approach (FS+LOESS):
  - Fit standard linear SEM model
  - Generate factor score estimates for latent predictor and outcome
  - Perform LOESS regression on factor score estimates
- Anticipated less bias for FM-SEM

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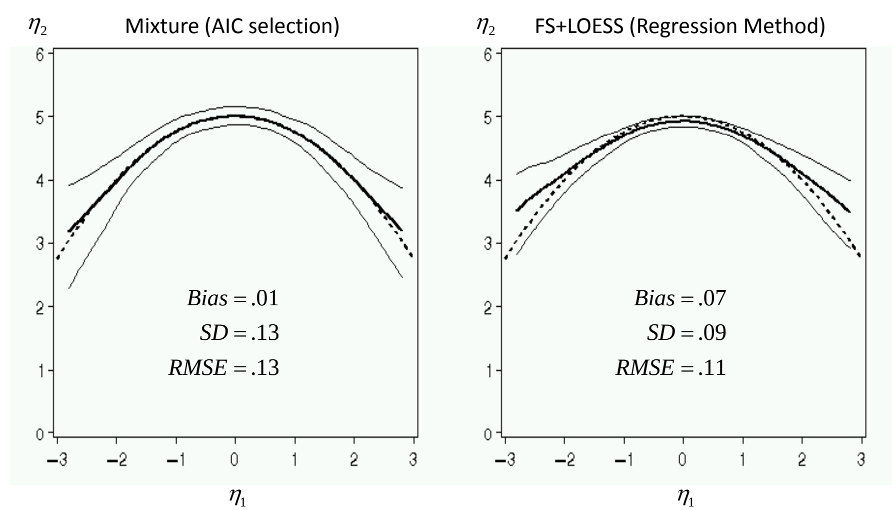

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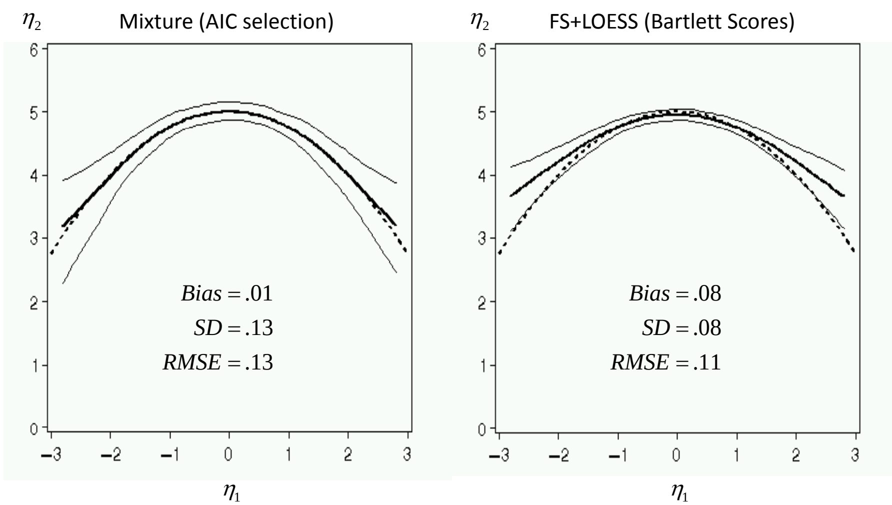

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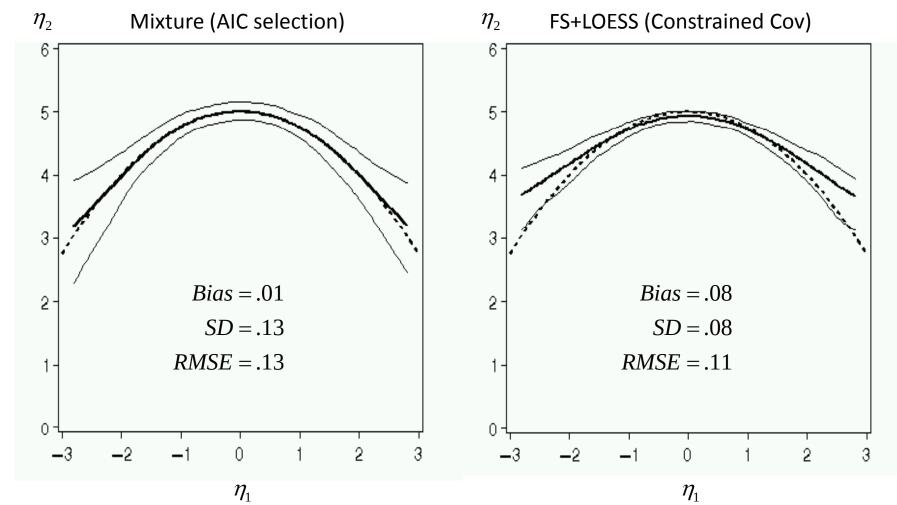

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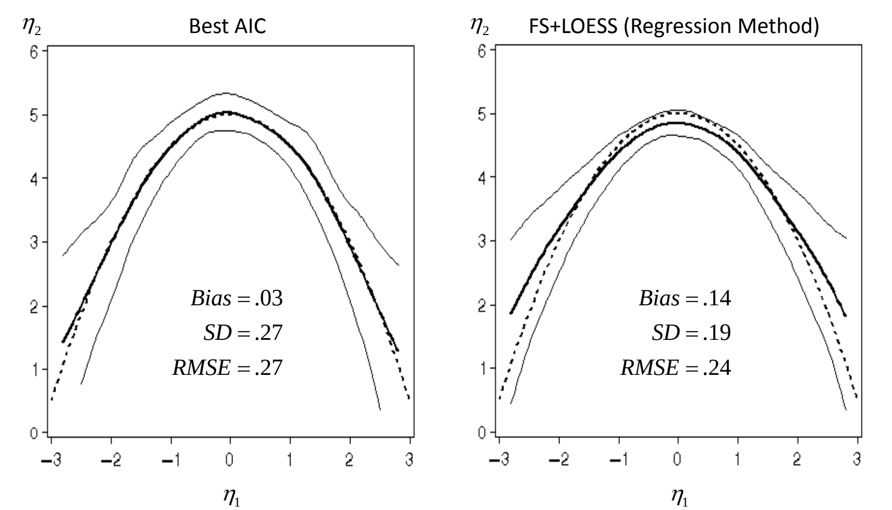

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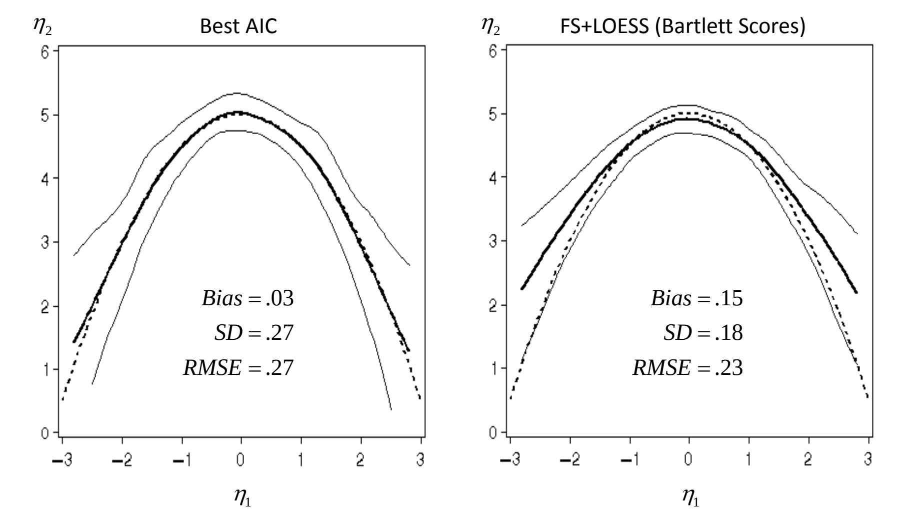

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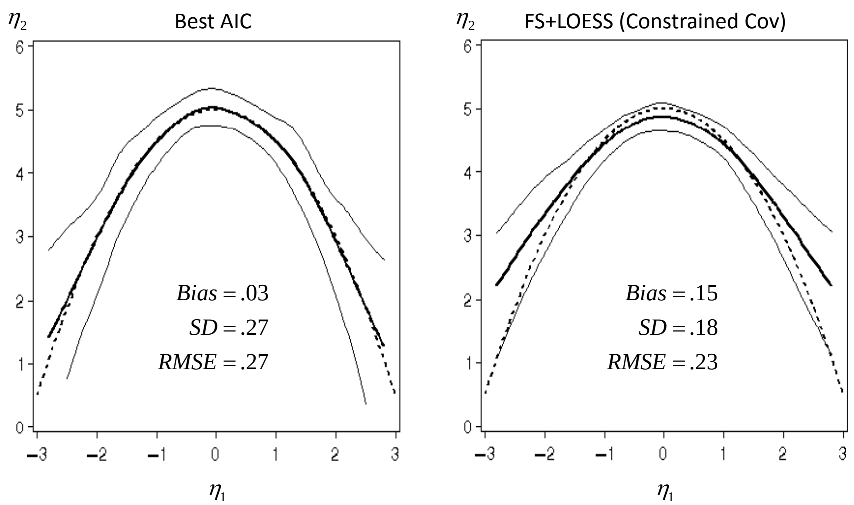
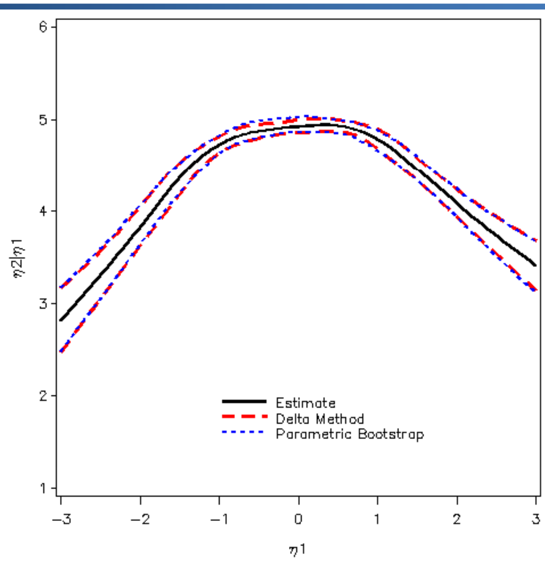


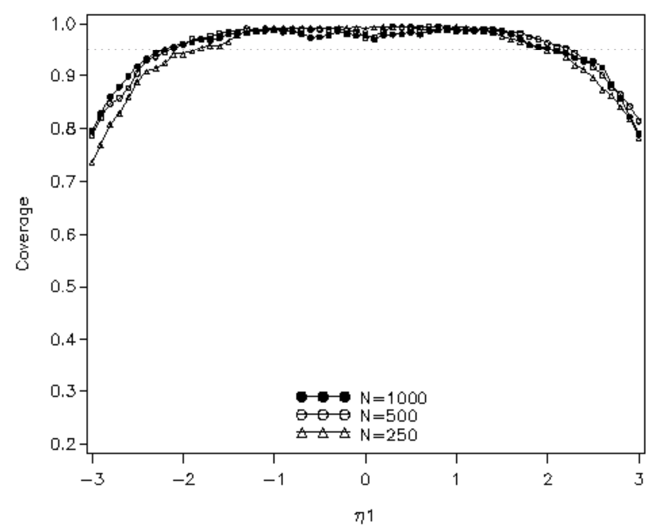
Can We Make Inferences?

- Since no *parameters* are estimated, inference must focus on *function*
- Monte Carlo plots have bands enclosing 95% of estimates across replications
- Similarly, want 95% confidence bands for any single replication
- Pointwise (non-simultaneous) confidence intervals constructed using two methods:
  - Delta Method
  - Parametric Bootstrap

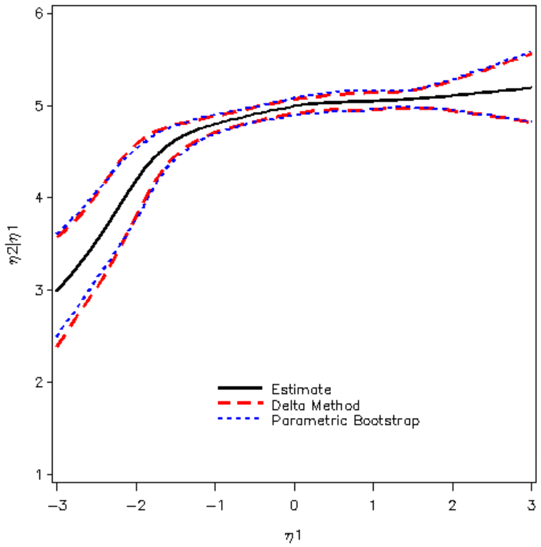
One Replication: Small Quadratic



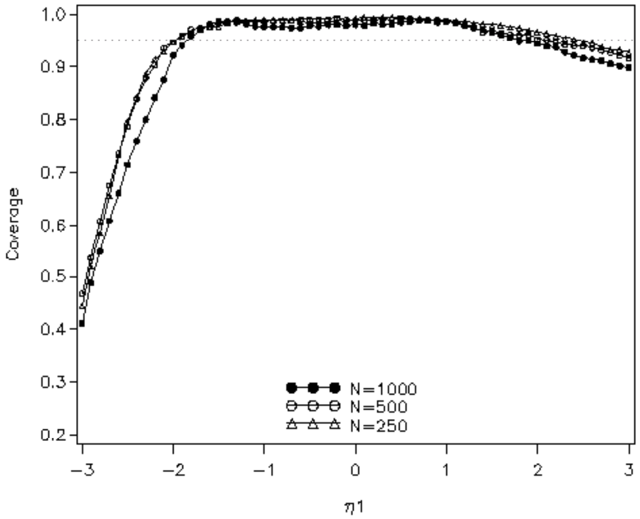
Coverage: Small Quadratic



One Replication: Exponential



Coverage: Exponential



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- *Where to next?*
  - Modeling nonlinear interactions for multiple latent predictors