Estimating Multilevel Linear Models as Structural Equation Models

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Outline

- **Describe Motivation**
- **Introduce Multilevel Linear Model (MLM)**
- **Show that MLM can be estimated as SEM**
- **Show that we can extend MLM within SEM**

Motivation

Stengths/Limitations of MLMs

- **Optimal For**
  - **Obtaining correct SEs for nested data**
  - **Estimating & predicting random effects**
- **Difficult For**
  - **Estimating measurement models**
  - **Obtaining explicit tests of mediation**

Strengths/Limitations of SEM

- **Opposite of above**

Goals are to combine the strengths of the two models and bridge modeling traditions

A 2-Level MLM w/ L1 Covariates

Level 1 Model

$$y_{ij} = \pi_{0j} + \sum_{p=1}^{P} \pi_{pj} x_{p_{ij}} + r_{ij}$$

where

$$r_{ij} \sim MVN(\mathbf{0}, \mathbf{\Sigma}_{r_j})$$

Level 2 Model

$$\pi_{0j} = \beta_{00} + u_{0j}$$

$$\pi_{1j} = \beta_{10} + u_{1j}$$

$$\vdots$$

$$\pi_{Pj} = \beta_{P0} + u_{Pj}$$

where

$$\pi_{0j}, \pi_{1j}, \cdots, \pi_{0P} \sim MVN(\boldsymbol{\beta}, \mathbf{T})$$

A 2-Level MLM w/ L1 Covariates

Reduced-Form Equation

$$y_{ij} = \underbrace{\beta_{00} + \sum_{p=1}^{P} \beta_{p0} x_{p_{ij}}}_{\text{Fixed Coefficients}} + \underbrace{u_{00} + \sum_{p=1}^{P} u_{pj} x_{p_{ij}}}_{\text{Random Coefficients}} + r_{ij}$$

This is a special case of the linear mixed-effects model of Laird & Ware (1982)

$$\mathbf{y}_j = \mathbf{X}_j \boldsymbol{\beta} + \mathbf{Z}_j \mathbf{u}_j + \mathbf{r}_j$$

where

$\mathbf{X}_j$ **is the design matrix for the fixed effects** $\boldsymbol{\beta}$

$\mathbf{Z}_j$ **is the design matrix for the random effects** $\mathbf{u}_j$

implying that

$$\mathbf{y}_j \sim MVN(\mathbf{X}_j \boldsymbol{\beta}, \mathbf{Z}_j \mathbf{T} \mathbf{Z}_j' + \boldsymbol{\Sigma}_{r_j})$$

From MLM to SEM

In our case, $\mathbf{X}_j = \mathbf{Z}_j$, so

$$\mathbf{y}_j \sim MVN(\mathbf{X}_j \boldsymbol{\beta}, \mathbf{X}_j \mathbf{T} \mathbf{X}_j' + \boldsymbol{\Sigma}_{r_j})$$

Further*, if the design is balanced* then $\mathbf{X}_j = \mathbf{X}$

and

$$\mathbf{y}_j \sim MVN(\mathbf{X} \boldsymbol{\beta}, \mathbf{X} \mathbf{T} \mathbf{X}' + \boldsymbol{\Sigma}_r)$$

This is the same structure as a CFA model where

$$\mathrm{X} = \Lambda$$
$$\beta = \kappa$$
$$\mathrm{T} = \Phi$$
$$\Sigma_r = \Theta_\delta$$

A Classic Case: The Growth Model

Multilevel Linear Growth Model

Level 1: $y_{ti} = \pi_{0i} + \pi_{1i} x_{ti} + r_{ti}$

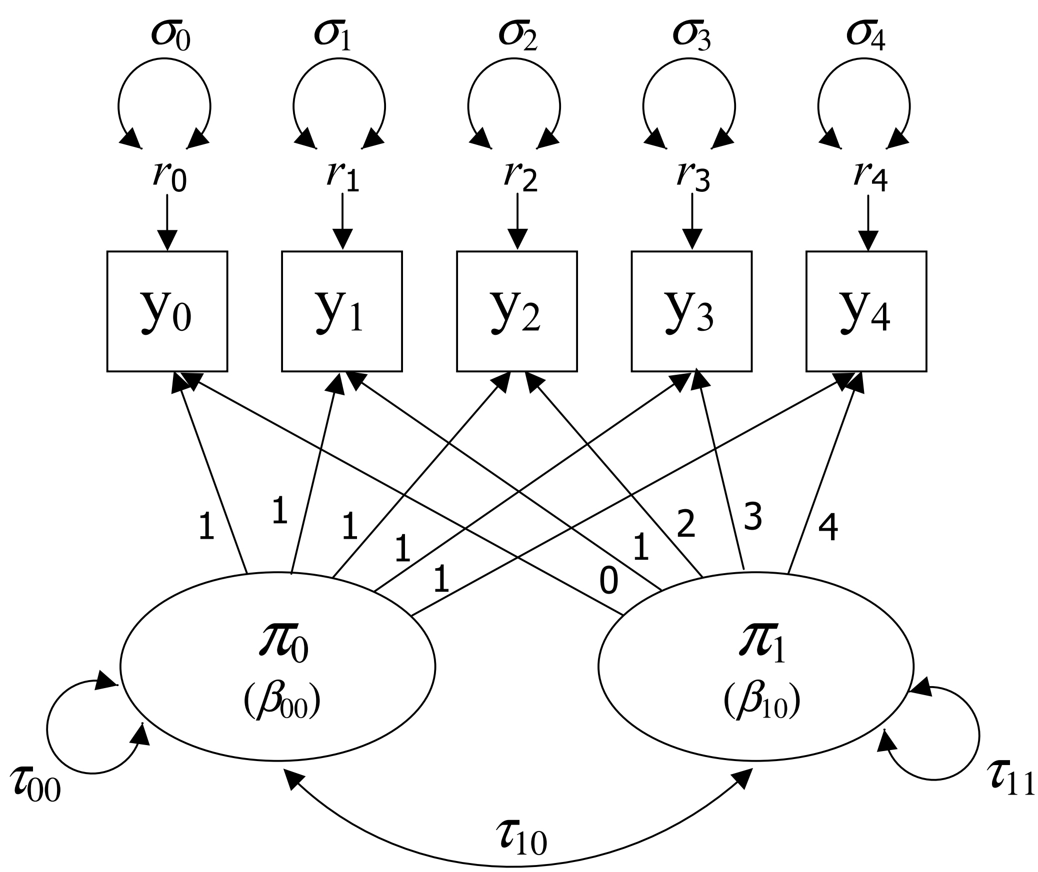
$$\Sigma_r = DIAG(\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4)$$

Level 2: $\pi_{0i} = \beta_{00} + u_{0i}$

$\pi_{1i} = \beta_{10} + u_{1i}$

$$T = \begin{pmatrix} \tau_{00} & \\ \tau_{10} & \tau_{11} \end{pmatrix}$$

Linear Latent Curve Model in SEM



$X_i = X = \Lambda$ because assuming balanced design.
Random coefficients are represented as latent variables.

Balanced Data

Example

3 male & 3 female students per school to evaluate effect of sex on language ability

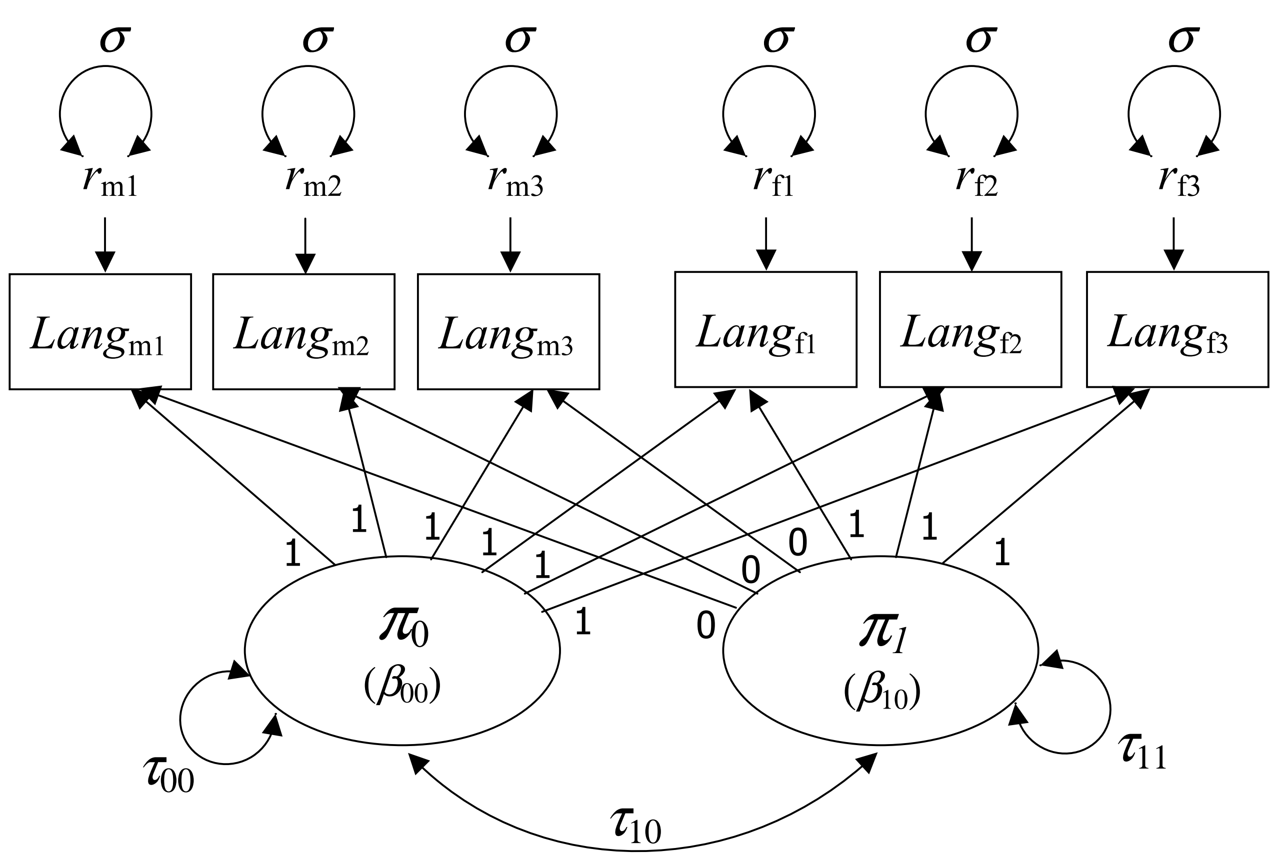
Multilevel Linear Model

Level 1: $lang_{ij} = \pi_{0j} + \pi_{1j} sex_{ij} + r_{ij}$ $\qquad \Sigma_r = \sigma I$

Level 2: $\pi_{0j} = \beta_{00} + u_{0j}$

$\pi_{1i} = \beta_{10} + u_{1j}$ $\qquad T = \begin{pmatrix} \tau_{00} & \\ \tau_{10} & \tau_{11} \end{pmatrix}$

Equivalent SEM



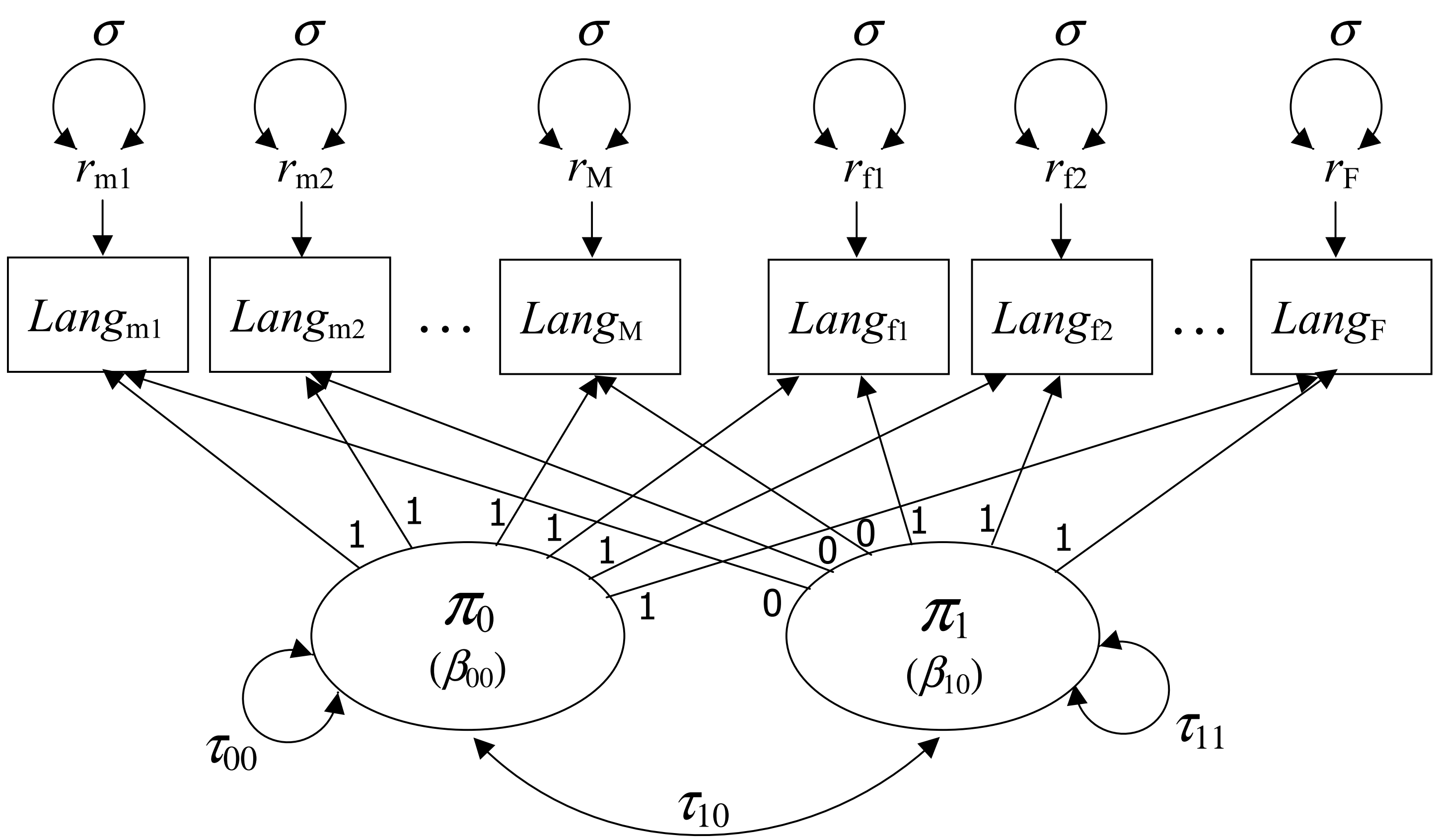
Order of the 3 males, 3 females w/in units j is arbitrary

Strategies for Imbalanced Data

Treat as missing

- **Construct complete-data** $\hat{\Sigma}(\theta), \hat{\mu}(\theta)$
- **Compare each** $\mathbf{y}_j$ **to submatrices** $\hat{\Sigma}(\theta)_j, \hat{\mu}(\theta)_j$

Example: **M = max # male students & F = max # female students in any given school.**

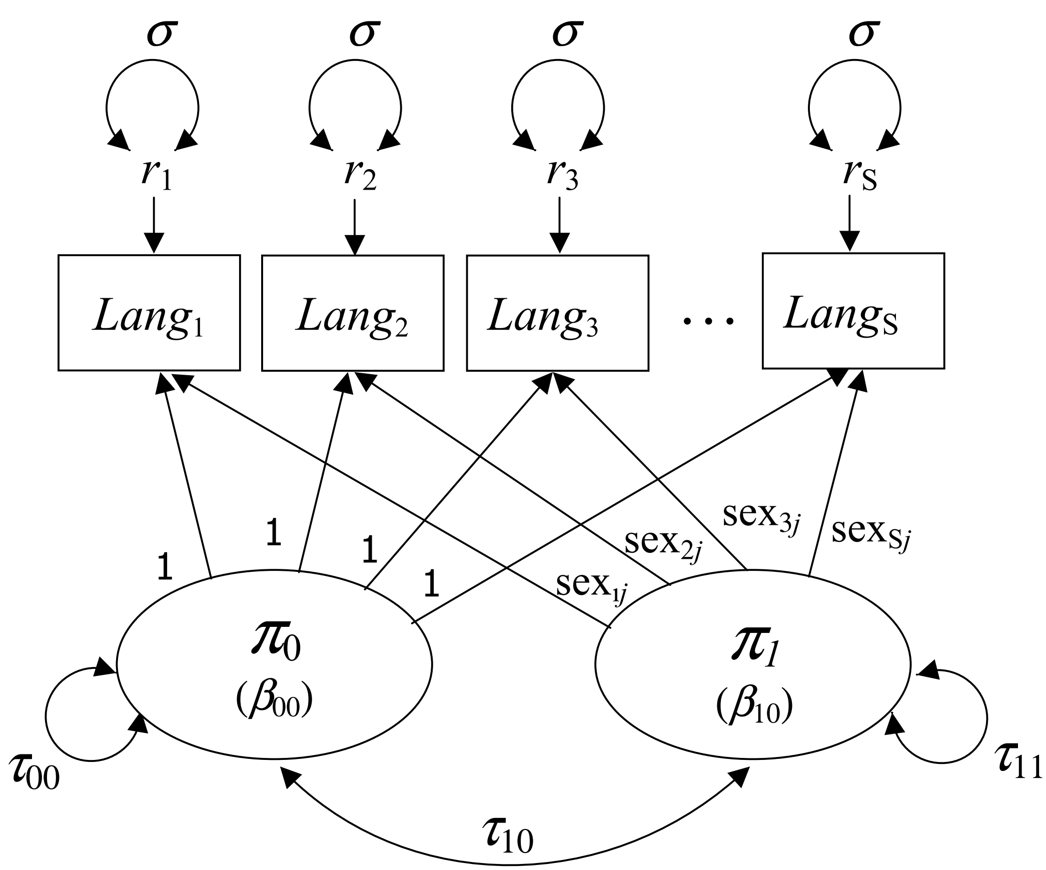


Strategies for Imbalanced Data

Compute $\hat{\Sigma}(\theta)_j, \hat{\mu}(\theta)_j$ ***directly from*** Λ_j

- **Truer to multilevel approach:** $\Lambda_j = \mathbf{X}_j$
- $\mathbf{X}_j$ **referred to as *definition variables* for** Λ_j
- **Due to Neale**

Example: **S = max # students in any given school**



What to do if you're imbalanced?

Both approaches provide computationally equivalent results but

- **Strategy 1 is better for few discrete covariates & complex residual structures.**
- **Strategy 2 is better for continuous covariates (highly imbalanced data) & homogeneity of error variance.**

Adding Higher-Level Predictors

Adding Level 2 Covariates

Problem is $\mathbf{X}_j \neq \mathbf{Z}_j$ but one Λ_j

Rovine & Molenaar Solution:

- **Fixed effects factors have means, no variance**
- **Random effects factors have variance, no means**
- **Define $\Lambda_j = BLOCK(\mathbf{X}_j, \mathbf{Z}_j)$**
- **True to mixed-effects model, non-intuitive.**

Alternative Solution:

- **Extends approach used w/ latent curve models**
- **L2 predictors are 'fixed X' covariates**
  - **Effects contained in Γ**
- **Computationally equivalent to R & M Solution**

Both solutions can be extended to 3+ Level models

Expanding the Model: A New Approach to Multilevel CFA

Adding a measurement model for item level outcomes

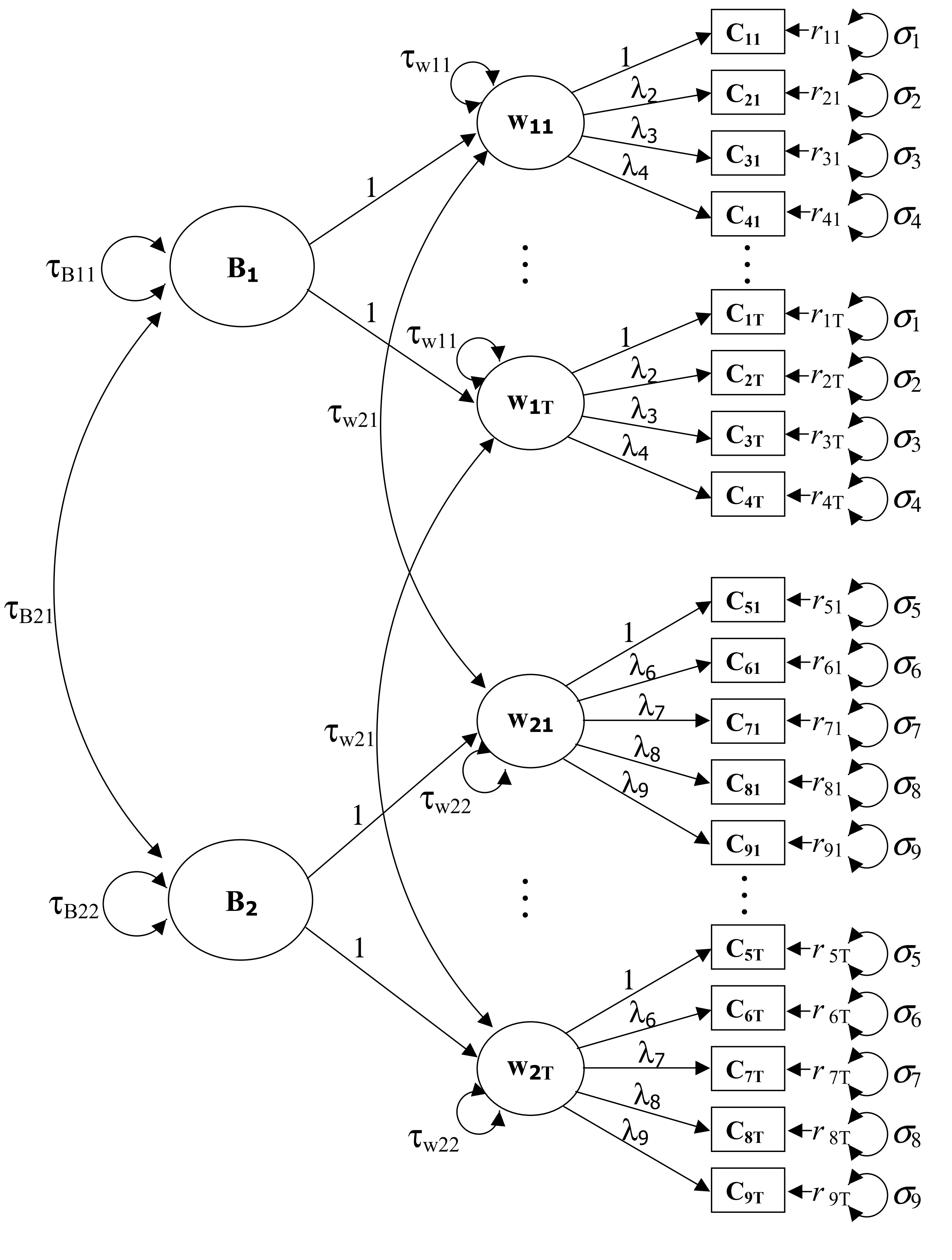
Example:

Data from High-School & Beyond: Teacher Survey

- **456 schools; 10,365 teachers**
  - **Imbalanced: # teachers/school ranges from 1 to 30**
  - **Let max # teachers = T = 30**
- **9 item measure of teacher perceptions of control**
  - **4 items on control of school policy**
  - **5 items on control of classroom teaching/planning**
  - **6 point Likert scales; Centered at mean**

Estimating 2-Factor Model

High-School & Beyond 2-Factor Model



Empirical Validation

Comparing SEM and MLM estimates

Parameter	Multilevel CFA		PROC MIXED	
λ_1	1.0		1.0	Loadings fixed to values from multilevel CFA (Cannot be estimated directly)
λ_2	.9932	(.02342)	.9932	
λ_3	1.1492	(.02508)	1.1492	
λ_4	1.2867	(.02615)	1.2867	
λ_5	1.0		1.0	
λ_6	.9803	(.01746)	.9803	
λ_7	.5444	(.01088)	.5444	
λ_8	.6080	(.01532)	.6080	
λ_9	.4269	(.01047)	.4269	
τ_{w11}	.5114	(.01848)	.5114	(.01132)
τ_{w22}	.6384	(.01970)	.6384	(.01292)
τ_{w21}	.2637	(.01011)	.2637	(.00885)
τ_{B11}	**.2029**	(.01726)	**.2027**	(.01647)
τ_{B22}	.1611	(.01426)	.1611	(.01379)
τ_{B21}	.1153	(.01240)	.1153	(.01232)
σ_1	1.2579	(.02160)	1.2579	(.02084)
σ_2	1.4890	(.02471)	1.4890	(.02413)
σ_3	1.3828	(.02481)	1.3828	(.02367)
σ_4	1.0047	(.02214)	1.0047	(.02015)
σ_5	.9614	(.01778)	.9614	(.01686)
σ_6	.5799	(.01295)	.5799	(.01176)
σ_7	.3675	(.00636)	.3675	(.00615)
σ_8	1.0119	(.01566)	1.0119	(.01548)
σ_9	.4610	(.00718)	.4610	(.00711)