# Comparing the Estimates of Mixed Models for Binary or Ordinal Data

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## Mixed Models

- **Mixed models provide a model-based way to account for dependence in clustered or longitudinal data.**
- **Dependence is usually accommodated through the introduction of random effects.**
  - **Random intercepts to account for mean differences between units.**

## The Linear Mixed Model

- **The general form of the linear mixed model is**

$$y_{ij} = \mathbf{x}_{ij}'\boldsymbol{\beta} + \mathbf{z}_{ij}'\mathbf{u}_j + r_{ij}$$

  - **observation $i$ is nested within unit $j$.**
  - $\mathbf{x}_{ij}$ **is a vector of predictors with fixed effects.**
  - $\boldsymbol{\beta}$ **is the vector of fixed effects.**
  - $\mathbf{z}_{ij}$ **is a vector of predictors with random effects.**
  - $\mathbf{u}_j$ **is the vector of random effects for unit $j$.**
  - $r_{ij}$ **is the observation-specific residual.**
- **Typically assume that**

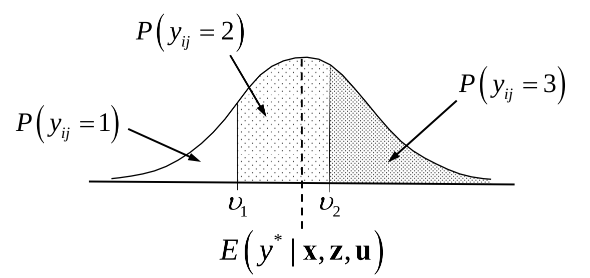
$$\begin{pmatrix}\mathbf{u}\\ \mathbf{r}\end{pmatrix} \sim N\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix}, \begin{bmatrix}\boldsymbol{\Sigma}_{\mathbf{u}} & 0\\ 0 & \sigma_r^2\mathbf{I}_n\end{bmatrix}\right)$$

## Mixed Model for Binary and Ordinal Outcomes

- **Assume that $y$ represents a coarse version of a continuous underlying variable $y^*$**

$$y_{ij}^* = \mathbf{x}_{ij}'\boldsymbol{\beta} + \mathbf{z}_{ij}'\mathbf{u}_j + r_{ij}$$

$$y_{ij} = c \text{ if } \upsilon^{c-1} < y_{ij}^* \le \upsilon^c \qquad (\upsilon_0 \equiv -\infty;\ \upsilon_C \equiv \infty)$$



## Scaling for Binary and Ordinal Outcomes

- **Because $y^*$ is not directly observed, its location and scale are arbitrary.**
- **Set the location by fixing first threshold: $\upsilon_1 \equiv 0$**
- **Set the scale by fixing $\sigma_r^2$**
  - **In the probit model, assume $r_{ij} \sim N(0,1)$**
  - **In the logit model, assume $r_{ij} \sim \text{logistic}\left(0, \frac{\pi^2}{3}\right)$**

## Comparing Sequential Models

- **Can we compare the estimates of sequential models?**
  - **Model 1: $y_{ij}^* = \beta_0 + u_j + r_{ij}$**
  - **Model 2: $y_{ij}^* = \beta_0 + \beta_1 x_{1ij} + u_j + r_{ij}$**
  - **Model 3: $y_{ij}^* = \beta_0 + \beta_1 x_{1ij} + \beta_2 x_{2ij} + u_j + r_{ij}$**
- **Across these models, $\sigma_r^2$ remains constant.**
  - **In the probit model: $\sigma_r^2 = 1$**
  - **In the logit model: $\sigma_r^2 = \pi^2 / 3$**
- **Variance of $y^*$ must then change between models, putting estimates on different scales.**

## Placing Estimates on a Commensurate Scale

- **To compare the estimates of simple logistic or probit models, Winship & Mare (1983, 1984) recommended equating the model-implied marginal variance of $y^*$ across the fitted models.**

## Placing Estimates on a Commensurate Scale

- **The model-implied marginal variance of $y^*$ in the mixed model is**

$$V\left(y_{ij}^*\right) = \boldsymbol{\beta}'\boldsymbol{\Sigma}_{\mathbf{x}}\boldsymbol{\beta} + \boldsymbol{\mu}_{\mathbf{z}}'\boldsymbol{\Sigma}_{\mathbf{u}}\boldsymbol{\mu}_{\mathbf{z}} + VEC\left(\boldsymbol{\Sigma}_{\mathbf{z}}\right)' VEC\left(\boldsymbol{\Sigma}_{\mathbf{u}}\right) + \sigma_r^2$$

- **It then follows that**

$$\left(\boldsymbol{\beta}'\boldsymbol{\Sigma}_{\mathbf{x}}\boldsymbol{\beta} + \boldsymbol{\mu}_{\mathbf{z}}'\boldsymbol{\Sigma}_{\mathbf{u}}\boldsymbol{\mu}_{\mathbf{z}} + VEC\left(\boldsymbol{\Sigma}_{\mathbf{z}}\right)' VEC\left(\boldsymbol{\Sigma}_{\mathbf{u}}\right) + \sigma_r^2\right)^{-1} V\left(y_{ij}^*\right) = 1$$

- **To rescale the implied variance to equal $a$, note that**

$$a\left(\boldsymbol{\beta}'\boldsymbol{\Sigma}_{\mathbf{x}}\boldsymbol{\beta} + \boldsymbol{\mu}_{\mathbf{z}}'\boldsymbol{\Sigma}_{\mathbf{u}}\boldsymbol{\mu}_{\mathbf{z}} + VEC\left(\boldsymbol{\Sigma}_{\mathbf{z}}\right)' VEC\left(\boldsymbol{\Sigma}_{\mathbf{u}}\right) + \sigma_r^2\right)^{-1} V\left(y_{ij}^*\right) = a$$

## Placing Estimates on a Commensurate Scale

- **Recall that** $s^2 V(y_{ij}^*) = V(sy_{ij}^*)$**, where** $s$ **is a scaling factor**

$$\underbrace{a\left(\boldsymbol{\beta}'\boldsymbol{\Sigma}_{\mathbf{x}}\boldsymbol{\beta} + \boldsymbol{\mu}_{\mathbf{z}}'\boldsymbol{\Sigma}_{\mathbf{u}}\boldsymbol{\mu}_{\mathbf{z}} + VEC\left(\boldsymbol{\Sigma}_{\mathbf{z}}\right)' VEC\left(\boldsymbol{\Sigma}_{\mathbf{u}}\right) + \sigma_r^2\right)^{-1}}_{s^2} V\left(y_{ij}^*\right) = a$$

- **We can apply this to the mixed model equation as follows:**

$$s = a^{1/2}\left(\boldsymbol{\beta}'\boldsymbol{\Sigma}_{\mathbf{x}}\boldsymbol{\beta} + \boldsymbol{\mu}_{\mathbf{z}}'\boldsymbol{\Sigma}_{\mathbf{u}}\boldsymbol{\mu}_{\mathbf{z}} + VEC\left(\boldsymbol{\Sigma}_{\mathbf{z}}\right)' VEC\left(\boldsymbol{\Sigma}_{\mathbf{u}}\right) + \sigma_r^2\right)^{-1/2}$$

$$sy_{ij}^* = \mathbf{x}_{ij}'\left(s\boldsymbol{\beta}\right) + \mathbf{z}_{ij}'\left(s\mathbf{u}_j\right) + \left(sr_{ij}\right)$$

$$VAR\left(sy_{ij}^*\right) = a$$

## Comparing Fitted Models

- **Given the equation**

$$sy_{ij}^* = \mathbf{x}_{ij}'\left(s\boldsymbol{\beta}\right) + \mathbf{z}_{ij}'\left(s\mathbf{u}_j\right) + \left(sr_{ij}\right)$$

**the rescaled fixed effects are** $s\boldsymbol{\beta}$

**the rescaled covariance parameters are** $s^2\boldsymbol{\Sigma}_{\mathbf{u}}$ **and** $s^2\sigma_r^2$

- **For any two models to be compared, can rescale estimates using a common** $a$ **value so that the units are commensurate.**

## Simulated Data Example

- **10,000 Clusters, 1 to 10 observations per cluster**
- **Population Model**

$$y_{ij}^* = \beta_0 + \beta_1 x_{1ij} + \beta_2 x_{2ij} + u_j + r_{ij} \qquad y = 1 \text{ if } y_{ij}^* > 0;\ \text{else } y = 0$$

$$\beta_0 = 0;\ \beta_1 = \beta_2 = 1$$

$$\sigma_u^2 = .05;\ \sigma_r^2 = .15;\ r_{ij} \sim N$$

$$\boldsymbol{\mu}_{\mathbf{x}} = \mathbf{0}$$

$$\boldsymbol{\Sigma}_{\mathbf{x}} = \begin{pmatrix} .1+.3 & 0 \\ 0 & .1+.3 \end{pmatrix}$$

$CORR(x_1, x_2) = 0$

## Sequence of Models

- **Fit two mixed-effects probit models to** $y$
  - **Model 1:** $P(y_{ij} = 1) = g\left(\beta_0 + \beta_1 x_{1ij} + u_j\right)$
  - **Model 2:** $P(y_{ij} = 1) = g\left(\beta_0 + \beta_1 x_{1ij} + \beta_2 x_{2ij} + u_j\right)$

  $g^{-1}$ **chosen to be probit link function.**

- **For comparison, fit two linear mixed models to simulated** $y^*$
  - **Model 1:** $y_{ij}^* = \beta_0 + \beta_1 x_{1ij} + u_j + r_{ij}$
  - **Model 2:** $y_{ij}^* = \beta_0 + \beta_1 x_{1ij} + \beta_2 x_{2ij} + u_j + r_{ij}$

*Estimates for models fit to simulated data.*

| *Effects* | *Fit to $y^*$* Linear | *Fit to y* Probit | Rescaled (a=1) | Rescaled (a=2) |
|---|---|---|---|---|
| *Model 1* | | | | |
| $\beta_0$ | 0.01 | 0.01 | 0.01 | 0.01 |
| $\beta_1$ | 1.02 | 1.52 | 1.03 | 1.45 |
| $\sigma_u^2$ | 0.15 | 0.29 | 0.13 | 0.26 |
| $\sigma_r^2$ | 0.44 | 1.00 | 0.46 | 0.91 |
| *Model 2* | | | | |
| $\beta_0$ | 0.00 | 0.01 | 0.00 | 0.01 |
| $\beta_1$ | 1.00 | 2.61 | 1.00 | 1.42 |
| $\beta_2$ | 0.99 | 2.62 | 1.00 | 1.42 |
| $\sigma_u^2$ | 0.05 | 0.30 | 0.04 | 0.09 |
| $\sigma_r^2$ | 0.15 | 1.00 | 0.15 | 0.29 |

## Conclusions

- **For continuous outcomes, often recommended to fit and compare sequentially fit models.**
- **For binary and ordinal outcomes, sequentially fit models cannot be meaningfully compared unless the estimates are rescaled to equate the model-implied variance of $y^*$.**

## Of Additional Interest…

- **Attention to scaling also clarifies several other issues that arise with mixed models for binary/ordinal outcomes:**
  - **Why marginal and conditional model estimates differ (inclusion of random effects changes scale)**
  - **Why estimators (e.g., MQL, PQL) that produce biased variance component estimates also produce biased fixed effects estimates (on wrong scale)**
  - **Why misspecification of the variance component structure can bias the fixed effects estimates (on wrong scale)**