1 Appendix A

The following level-1 model is assumed to hold for the jth ISU:

$$Y_{ij} = \beta_{0j} + \beta_{1j} X_{ij} + \beta_{2j} X_{ij}^2 + r_{ij}, \tag{1}$$

and the level-2 equations are:

$$\begin{aligned} \beta_{0j} &= \gamma_{00} + u_{0j}, \\ \beta_{1j} &\equiv \gamma_{10}, \\ \beta_{2j} &\equiv \gamma_{20}, \end{aligned}$$

where $u_{0j} \sim N(0, \tau_{00})$, which implies $\beta_{0j} \sim N(\gamma_{00}, \tau_{00})$, and the r_{ij}'s are independent $N(0, \sigma)$ variates. The combined mixed model equation is

$$Y_{ij} = \gamma_{00} + \gamma_{10} X_{ij} + \gamma_{20} X_{ij}^2 + u_{0j} + r_{ij}. \tag{2}$$

The predictor is assumed to be generated in the following way:

$$X_{ij} = \mu_j + e_{ij}, \tag{3}$$

where e_{ij}'s are independent $N(0, \phi)$ variates, and $\mu_j \sim N(\mu, \psi)$. Furthermore, e_{ij}, r_{ij}, u_{0j}, and μ_j are mutually independent. The foregoing implies the following:

$$E(e_{ij}) = 0, E(e_{ij}^2) = \phi, E(e_{ij}^3) = 0.$$
$$E(\mu_j) = \mu, E(\mu_j^2) = \psi + \mu^2, E(\mu_j^3) = \mu^3 + 3\mu\psi, E(\mu_j^4) = \mu^4 + 6\mu^2\psi + 3\psi^2.$$

Suppose that a two-level model is fitted with level-1 model:

$$Y_{ij} = \beta_{0j}^* + \beta_{1j}^* X_{ij} + r_{ij}^*, \tag{4}$$

and level-2 model:

$$\begin{aligned} \beta_{0j}^* &= \gamma_{00}^* + u_{0j}^*, \\ \beta_{1j}^* &= \gamma_{10}^* + u_{1j}^*, \end{aligned}$$

where $E(r_{ij}^*) = 0$, $VAR(r_{ij}^*) = \sigma^*$, $E(u_{0j}^*) = 0$, $VAR(u_{0j}^*) = \tau_{00}^*$, $E(u_{1j}^*) = 0$, $VAR(u_{1j}^*) = \tau_{11}^*$, and $COV(u_{0j}^*, u_{1j}^*) = \tau_{01}^*$. The mixed model equation for this two-level model is

$$Y_{ij} = \gamma_{00}^* + \gamma_{10}^* X_{ij} + u_{0j}^* + u_{1j}^* X_{ij} + r_{ij}^*. \tag{5}$$

To find the fixed-effects coefficients and (co)variance components, it suffices to find the individual regression coefficients β_{0j} and β_{1j}, and then take expectations. Note that

$$\begin{aligned} E(X_{ij}|\mu_j) &= \mu_j, \\ E(X_{ij}^2|\mu_j) &= E(e_{ij}^2 + \mu_j^2 + 2e_{ij}\mu_j|\mu_j) = \phi + \mu_j^2, \\ E(X_{ij}^3|\mu_j) &= E(e_{ij}^3 + \mu_j^3 + 3e_{ij}^2\mu_j + 3e_{ij}\mu_j^2|\mu_j) = \mu_j^3 + 3\mu_j\phi, \end{aligned}$$

and that

$$\begin{aligned} E(Y_{ij}X_{ij}|\mu_j) &= E(\beta_{0j}X_{ij} + \gamma_{10}X_{ij}^2 + \gamma_{20}X_{ij}^3 + r_{ij}X_{ij}|\mu_j) \\ &= \beta_{0j}\mu_j + \gamma_{10}(\phi + \mu_j^2) + \gamma_{20}(\mu_j^3 + 3\mu_j\phi), \\ E(Y_{ij}|\mu_j) &= E(\beta_{0j} + \gamma_{10}X_{ij} + \gamma_{20}X_{ij}^2 + r_{ij}|\mu_j) \\ &= \beta_{0j} + \gamma_{10}\mu_j + \gamma_{20}(\phi + \mu_j^2). \end{aligned}$$

Therefore,

$$\begin{aligned} COV(Y_{ij}, X_{ij}|\mu_j) &= E(Y_{ij}X_{ij}|\mu_j) - E(Y_{ij}|\mu_j)E(X_{ij}|\mu_j) \\ &= \gamma_{10}\phi + 2\gamma_{20}\phi\mu_j. \end{aligned}$$

This implies that

$$\begin{aligned} \beta^*_{1j} &= \frac{COV(Y_{ij}, X_{ij}|\mu_j)}{VAR(X_{ij}|\mu_j)} \qquad (6) \\ &= \frac{\gamma_{10}\phi + 2\gamma_{20}\phi\mu_j}{\phi + \mu_j^2 - \mu_j^2} = \gamma_{10} + 2\gamma_{20}\mu_j, \\ \beta^*_{0j} &= E(Y_{ij}|\mu_j) - \beta^*_{1j}E(X_{ij}|\mu_j) \qquad (7) \\ &= \beta_{0j} + \gamma_{10}\mu_j + \gamma_{20}(\phi + \mu_j^2) - (\gamma_{10} + 2\gamma_{20}\mu_j)\mu_j \\ &= \beta_{0j} + \gamma_{20}(\phi - \mu_j^2). \end{aligned}$$

This implies that

$$\begin{aligned} \gamma^*_{00} &= E(\beta^*_{0j}) = \gamma_{00} + \gamma_{20}(\phi - \psi - \mu^2), \\ \gamma^*_{10} &= E(\beta^*_{1j}) = \gamma_{10} + 2\gamma_{20}\mu, \\ \tau^*_{00} &= VAR(\beta^*_{0j}) = \tau_{00} + \gamma_{20}^2(4\mu^2\psi + 2\psi^2), \\ \tau^*_{11} &= VAR(\beta^*_{1j}) = 4\gamma_{20}^2\psi. \end{aligned}$$

In addition,

$$\begin{aligned} E(\beta^*_{1j}\beta^*_{0j}) &= E(\gamma_{10}\beta_{0j} + \gamma_{10}\gamma_{20}\phi - \gamma_{10}\gamma_{20}\mu_j^2 + 2\gamma_{20}\mu_j\beta_{0j} + 2\gamma_{20}^2\mu_j\phi - 2\gamma_{20}^2\mu_j^3) \\ &= \gamma_{10}\gamma_{00} + \gamma_{10}\gamma_{20}\phi - \gamma_{10}\gamma_{20}(\psi + \mu^2) + 2\gamma_{20}\mu\gamma_{00} + 2\gamma_{20}^2\mu\phi - 2\gamma_{20}^2(\mu^3 + 3\mu\psi), \\ &= \gamma_{00}\gamma_{10} + \gamma_{10}\gamma_{20}\phi - \gamma_{10}\gamma_{20}\psi - \gamma_{10}\gamma_{20}\mu^2 + 2\gamma_{00}\gamma_{20}\mu + 2\gamma_{20}^2\phi\mu - 2\gamma_{20}^2\mu^3 - 6\gamma_{20}^2\psi\mu, \\ E(\beta^*_{1j})E(\beta^*_{0j}) &= \gamma_{00}\gamma_{10} + \gamma_{10}\gamma_{20}\phi - \gamma_{10}\gamma_{20}\psi - \gamma_{10}\gamma_{20}\mu^2 + 2\gamma_{00}\gamma_{20}\mu + 2\gamma_{20}^2\phi\mu - 2\gamma_{20}^2\mu^3 - 2\gamma_{20}^2\psi\mu \end{aligned}$$

implies $\tau^*_{01} = COV(\beta^*_{1j}, \beta^*_{0j}) = -4\gamma_{20}^2\psi\mu$.

2 Appendix B

The same level-1 model in Equation (1) continues to hold, but there is an additional level-2 predictor W_j:

$$\begin{aligned}\beta_{0j} &= \gamma_{00} + \gamma_{01} W_j + u_{0j},\\ \beta_{1j} &\equiv \gamma_{10},\\ \beta_{2j} &\equiv \gamma_{20},\end{aligned}$$

where $W_j \sim N(\nu, \lambda)$ is uncorrelated with u_{0j}, and $CORR(\mu_j, W_j) = \rho$, i.e., $COV(\mu_j, W_j) = \rho\sqrt{\psi\lambda}$. All the other assumptions continue to hold. The generating model's mixed model equation is

$$Y_{ij} = \gamma_{00} + \gamma_{01} W_j + \gamma_{10} X_{ij} + \gamma_{20} X_{ij}^2 + u_{0j} + r_{ij}. \tag{8}$$

Consider fitting the following level-1 model:

$$Y_{ij} = \beta_{0j}^* + \beta_{1j}^* X_{ij} + r_{ij}^*, \tag{9}$$

and level-2 model:

$$\begin{aligned}\beta_{0j}^* &= \gamma_{00}^* + \gamma_{01}^* W_j + u_{0j}^*,\\ \beta_{1j}^* &= \gamma_{10}^* + \gamma_{11}^* W_j + u_{1j}^*,\end{aligned}$$

which corresponds to the following mixed model equation

$$Y_{ij} = \gamma_{00}^* + \gamma_{01}^* W_j + \gamma_{10}^* X_{ij} + \gamma_{11}^* W_j X_{ij} + u_{0j}^* + u_{1j}^* X_{ij} + r_{ij}^*. \tag{10}$$

Using existing results in (6) and (7), we find

$$COV(\beta_{1j}^*, W_j) = COV(\gamma_{10} + 2\gamma_{20}\mu_j, W_j) = 2\gamma_{20}\rho\sqrt{\psi\lambda},$$

which means

$$\gamma_{11}^* = \frac{COV(\beta_{1j}^*, W_j)}{VAR(W_j)} = \frac{2\gamma_{20}\rho\sqrt{\psi}}{\sqrt{\lambda}},$$

and

$$\gamma_{10}^* = \gamma_{10} + 2\gamma_{20}\left(\mu - \frac{\nu\rho\sqrt{\psi}}{\sqrt{\lambda}}\right).$$

To find γ_{00} and γ_{01}, note that

$$COV(\mu_j^2, W_j) = 2\rho\sqrt{\psi\lambda}\mu,$$

so that

$$\begin{aligned}COV(\beta_{0j}^*, W_j) &= COV(\beta_{0j} + \gamma_{20}\phi - \gamma_{20}\mu_j^2, W_j)\\ &= COV(\beta_{0j}, W_j) - \gamma_{20} COV(\mu_j^2, W_j)\\ &= COV(\gamma_{00} + \gamma_{01} W_j + u_{0j}, W_j) - \gamma_{20} COV(\mu_j^2, W_j)\\ &= \gamma_{01}\lambda - 2\gamma_{20}\rho\sqrt{\psi\lambda}\mu.\end{aligned}$$

Therefore,

$$\begin{aligned}\gamma_{01}^* &= \frac{COV(\beta_{0j}^*, W_j)}{VAR(W_j)}\\ &= \gamma_{01} - \frac{2\gamma_{20}\rho\sqrt{\psi}\mu}{\sqrt{\lambda}},\\ \gamma_{00}^* &= E(\beta_{0j}^*) - \gamma_{01}^* E(W_j)\\ &= \gamma_{00} + \gamma_{20}(\phi - \psi - \mu^2) - \gamma_{01}\nu + \frac{2\gamma_{20}\rho\sqrt{\psi}\mu\nu}{\sqrt{\lambda}}\end{aligned}$$

The residual variance of the random slope is also of interest. Since

$$\begin{aligned} u_{1j}^* &= \beta_{1j}^* - (\gamma_{10}^* + \gamma_{11}^* W_j) \\ &= 2\gamma_{20}\mu_j - 2\gamma_{20}\mu + \frac{2\gamma_{20}\nu\rho\sqrt{\psi}}{\sqrt{\lambda}} - \frac{2\gamma_{20}\rho\sqrt{\psi}}{\sqrt{\lambda}} W_j, \end{aligned}$$

we see that

$$\begin{aligned} \tau_{11}^* = VAR(u_{1j}^*) &= V(2\gamma_{20}\mu_j - \frac{2\gamma_{20}\rho\sqrt{\psi}}{\sqrt{\lambda}} W_j) \\ &= 4\gamma_{20}^2 VAR(\mu_j) + \left(\frac{2\gamma_{20}\rho\sqrt{\psi}}{\sqrt{\lambda}}\right)^2 VAR(W_j) - 4\gamma_{20}\frac{2\gamma_{20}\rho\sqrt{\psi}}{\sqrt{\lambda}} COV(\mu_j, W_j) \\ &= 4\gamma_{20}^2\psi(1-\rho^2). \end{aligned}$$