1 Appendix A

The following level-1 model is assumed to hold for the $j$th ISU:

$$Y_{ij} = \beta_{0j} + \beta_{1j} X_{ij} + \beta_{2j} X_{ij}^2 + r_{ij},$$

and the level-2 equations are:

$$\begin{align*}
\beta_{0j} &= \gamma_{00} + u_{0j}, \\
\beta_{1j} &= \gamma_{10}, \\
\beta_{2j} &= \gamma_{20},
\end{align*}$$

where $u_{0j} \sim N(0, \tau_{00})$, which implies $\beta_{0j} \sim N(\gamma_{00}, \tau_{00})$, and the $r_{ij}$’s are independent $N(0, \sigma)$ variates. The combined mixed model equation is

$$Y_{ij} = \gamma_{00} + \gamma_{10} X_{ij} + \gamma_{20} X_{ij}^2 + u_{0j} + r_{ij}. \tag{2}$$

The predictor is assumed to be generated in the following way:

$$X_{ij} = \mu_j + e_{ij}, \tag{3}$$

where $e_{ij}$’s are independent $N(0, \phi)$ variates, and $\mu_j \sim N(\mu, \psi)$. Furthermore, $e_{ij}, r_{ij}, u_{0j},$ and $\mu_j$ are mutually independent. The foregoing implies the following:

$$E(e_{ij}) = 0, E(e_{ij}^2) = \phi, E(e_{ij}^3) = 0.$$

$$E(\mu_j) = \mu, E(\mu_j^2) = \psi + \mu^2, E(\mu_j^3) = \mu^3 + 3\mu\psi, E(\mu_j^4) = \mu^4 + 6\mu^2\psi + 3\psi^2.$$

Suppose that a two-level model is fitted with level-1 model:

$$Y_{ij} = \beta_{0j}^* + \beta_{1j}^* X_{ij} + r_{ij}^*, \tag{4}$$

and level-2 model:

$$\begin{align*}
\beta_{0j}^* &= \gamma_{00}^* + u_{0j}^*, \\
\beta_{1j}^* &= \gamma_{10}^* + u_{1j}^*,
\end{align*}$$

where $E(r_{ij}^*) = 0$, $VAR(r_{ij}^*) = \sigma^*$, $E(u_{0j}^*) = 0$, $VAR(u_{0j}^*) = \tau_{00}^*$, $E(u_{1j}^*) = 0$, $VAR(u_{1j}^*) = \tau_{11}^*$, and $COV(u_{0j}^*, u_{1j}^*) = \tau_{01}^*$. The mixed model equation for this two-level model is

$$Y_{ij} = \gamma_{00}^* + \gamma_{10}^* X_{ij} + u_{0j}^* + u_{1j}^* X_{ij} + r_{ij}^*. \tag{5}$$

To find the fixed-effects coefficients and (co)variance components, it suffices to find the individual regression coefficients $\beta_{0j}$ and $\beta_{1j}$, and then take expectations. Note that

$$\begin{align*}
E(X_{ij}|\mu_j) &= \mu_j, \\
E(X_{ij}^2|\mu_j) &= E(e_{ij}^2 + \mu_j^2 + 2e_{ij}\mu_j|\mu_j) = \phi + \mu_j^2, \\
E(X_{ij}^3|\mu_j) &= E(e_{ij}^3 + \mu_j^3 + 3e_{ij}\mu_j + 3e_{ij}\mu_j^2|\mu_j) = \mu_j^3 + 3\mu_j\phi,
\end{align*}$$

and that

$$\begin{align*}
E(Y_{ij}|X_{ij}|\mu_j) &= E(\beta_{0j} X_{ij} + \gamma_{10} X_{ij}^2 + \gamma_{20} X_{ij}^3 + r_{ij} X_{ij}|\mu_j) \\
&= \beta_{0j} \mu_j + \gamma_{10}(\phi + \mu_j^2) + \gamma_{20}(\mu_j^3 + 3\mu_j\phi), \\
E(Y_{ij}|X_{ij}|\mu_j) &= E(\beta_{0j} + \gamma_{10} X_{ij} + \gamma_{20} X_{ij}^2 + r_{ij}|\mu_j) \\
&= \beta_{0j} + \gamma_{10}\mu_j + \gamma_{20}(\phi + \mu_j^2).
\end{align*}$$

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Therefore,

\[
\text{COV}(Y_{ij}, X_{ij}|\mu_j) = E(Y_{ij}X_{ij}|\mu_j) - E(Y_{ij}|\mu_j)E(X_{ij}|\mu_j)
= \gamma_{10}\phi + 2\gamma_{20}\phi\mu_j.
\]

This implies that

\[
\beta_{1j}^* = \frac{\text{COV}(Y_{ij}, X_{ij}|\mu_j)}{\text{VAR}(X_{ij}|\mu_j)} = \frac{\gamma_{10}\phi + 2\gamma_{20}\phi\mu_j}{\phi + \mu_j^2 - \mu_j^2} = \gamma_{10} + 2\gamma_{20}\mu_j, (6)
\]

\[
\beta_{0j}^* = E(Y_{ij}|\mu_j) - \beta_{1j}^*E(X_{ij}|\mu_j)
= \beta_{0j} + \gamma_{10}\mu_j + \gamma_{20}(\phi + \mu_j^2) - (\gamma_{10} + 2\gamma_{20}\mu_j)\mu_j
= \beta_{0j} + \gamma_{20}(\phi - \mu_j^2). (7)
\]

This implies that

\[
\gamma_{00}^* = E(\beta_{0j}^*) = \gamma_{00} + \gamma_{20}(\phi - \psi - \mu^2),
\]

\[
\gamma_{10}^* = E(\beta_{1j}^*) = \gamma_{10} + 2\gamma_{20}\mu,
\]

\[
\tau_{00}^* = \text{VAR}(\beta_{0j}^*) = \tau_{00} + \gamma_{20}^2(4\mu^2\psi + 2\psi^2),
\]

\[
\tau_{11}^* = \text{VAR}(\beta_{1j}^*) = 4\gamma_{20}^2\psi.
\]

In addition,

\[
E(\beta_{1j}^*\beta_{0j}^*) = E(\gamma_{10}\beta_{0j} + \gamma_{10}\gamma_{20}\phi - \gamma_{10}\gamma_{20}\mu_j^2 + 2\gamma_{20}\mu_j\beta_{0j} + 2\gamma_{20}^2\mu_j\phi - 2\gamma_{20}^2\mu_j^3)
= \gamma_{10}\gamma_{00} + \gamma_{10}\gamma_{20}\phi - \gamma_{10}\gamma_{20}(\psi + \mu^2) + 2\gamma_{20}\mu_j\gamma_{00} + 2\gamma_{20}^2\mu_j\phi - 2\gamma_{20}^2(\mu^3 + 3\mu\psi),
\]

\[
E(\beta_{1j}^*)E(\beta_{0j}^*) = \gamma_{00}\gamma_{10} + \gamma_{10}\gamma_{20}\phi - \gamma_{10}\gamma_{20}\psi - \gamma_{10}\gamma_{20}\mu^2 + 2\gamma_{00}\gamma_{20}\mu + 2\gamma_{20}^2\phi\mu - 2\gamma_{20}^2\mu^3 - 6\gamma_{20}^2\psi\mu,
\]

implies \(\tau_{01}^* = \text{COV}(\beta_{1j}^*, \beta_{0j}^*) = -4\gamma_{20}^2\psi\mu.\)
2 Appendix B

The same level-1 model in Equation (1) continues to hold, but there is an additional level-2 predictor $W_j$:

$$
\beta_{0j} = \gamma_{00} + \gamma_{01} W_j + u_{0j},
$$
$$
\beta_{1j} = \gamma_{10},
$$
$$
\beta_{2j} = \gamma_{20},
$$

where $W_j \sim N(\nu, \lambda)$ is uncorrelated with $u_{0j}$, and $\text{CORR}(\mu_j, W_j) = \rho$, i.e., $\text{COV}(\mu_j, W_j) = \rho \sqrt{\psi \lambda}$. All the other assumptions continue to hold. The generating model's mixed model equation is

$$
Y_{ij} = \gamma_{00} + \gamma_{01} W_j + \gamma_{10} X_{ij} + \gamma_{20} X_{ij}^2 + u_{0j} + r_{ij}.
$$

(8)

Consider fitting the following level-1 model:

$$
Y_{ij} = \beta_{0j}^* + \beta_{1j}^* X_{ij} + r_{ij}^*,
$$

(9)

and level-2 model:

$$
\beta_{0j}^* = \gamma_{00}^* + \gamma_{01}^* W_j + u_{0j}^*,
$$
$$
\beta_{1j}^* = \gamma_{10}^* + \gamma_{11}^* W_j + u_{1j}^*,
$$

which corresponds to the following mixed model equation

$$
Y_{ij} = \gamma_{00}^* + \gamma_{01}^* W_j + \gamma_{10}^* X_{ij} + \gamma_{11}^* W_j X_{ij} + u_{0j}^* + u_{1j}^* X_{ij} + r_{ij}^*.
$$

(10)

Using existing results in (6) and (7), we find

$$
\text{COV}(\beta_{1j}^*, W_j) = \text{COV}(\gamma_{10} + 2\gamma_{20} \mu_j, W_j) = 2\gamma_{20} \rho \sqrt{\psi \lambda},
$$

which means

$$
\gamma_{11}^* = \frac{\text{COV}(\beta_{1j}^*, W_j)}{\text{VAR}(W_j)} = \frac{2\gamma_{20} \rho \sqrt{\psi}}{\sqrt{\lambda}},
$$

and

$$
\gamma_{10}^* = \gamma_{10} + 2\gamma_{20} \left( \mu - \frac{\nu \rho \sqrt{\psi}}{\sqrt{\lambda}} \right).
$$

To find $\gamma_{00}$ and $\gamma_{01}$, note that

$$
\text{COV}(\mu_j^2, W_j) = 2\rho \sqrt{\psi \lambda} \mu,
$$

so that

$$
\text{COV}(\beta_{0j}^*, W_j) = \text{COV}(\beta_{0j} + 2\gamma_{20} \mu_j, W_j)
$$
$$
= \text{COV}(\beta_{0j}, W_j) - \gamma_{20} \text{COV}(\mu_j^2, W_j)
$$
$$
= \text{COV}(\gamma_{00} + \gamma_{01} W_j + u_{0j}, W_j) - \gamma_{20} \text{COV}(\mu_j^2, W_j)
$$
$$
= \gamma_{01} \lambda - 2\gamma_{20} \rho \sqrt{\psi \lambda} \mu.
$$

Therefore,

$$
\gamma_{01}^* = \frac{\text{COV}(\beta_{0j}^*, W_j)}{\text{VAR}(W_j)}
$$
$$
= \gamma_{01} - \frac{2\gamma_{20} \rho \sqrt{\psi \mu}}{\sqrt{\lambda}},
$$
$$
\gamma_{00}^* = E(\beta_{0j}^*) - \gamma_{01}^* E(W_j)
$$
$$
= \gamma_{00} + \gamma_{20} (\phi - \psi - \mu^2) - \gamma_{01} \nu + \frac{2\gamma_{20} \rho \sqrt{\psi \mu \nu}}{\sqrt{\lambda}}.
$$
The residual variance of the random slope is also of interest. Since

\[ u_{1j}^* = \beta_{1j}^* - (\gamma_{10}^* + \gamma_{11}^* W_j) \]

\[ = 2\gamma_{20} \mu_j - 2\gamma_{20} \mu + \frac{2\gamma_{20} \nu \rho \sqrt{\psi}}{\sqrt{\lambda}} - \frac{2\gamma_{20} \rho \sqrt{\psi}}{\sqrt{\lambda}} W_j, \]

we see that

\[ \tau_{11}^* = \text{VAR}(u_{1j}^*) = V(2\gamma_{20} \mu_j - \frac{2\gamma_{20} \rho \sqrt{\psi}}{\sqrt{\lambda}} W_j) \]

\[ = 4\gamma_{20}^2 \text{VAR}(\mu_j) + \left( \frac{2\gamma_{20} \rho \sqrt{\psi}}{\sqrt{\lambda}} \right)^2 \text{VAR}(W_j) - 4\gamma_{20}^2 \frac{2\gamma_{20} \rho \sqrt{\psi}}{\sqrt{\lambda}} \text{COV}(\mu_j, W_j) \]

\[ = 4\gamma_{20}^2 \psi (1 - \rho^2). \]